Equilibrium

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Türevin Yorumu

Birinci Türev

- Birinci ve ikinci türevlerinin verdiği bilgilerden f'(x) veya df/dx olarak yazılan f(x) fonksiyonunun ilk türevi, x noktasındaki teğet çizgisi fonksiyonun eğimidir.

- Grafik olmayan terimlerle ifade etmek gerekirse, ilk türev bize bir fonksiyonun nasıl arttığını veya azaldığını ve ne kadar artacağını veya azalacağını söyler.

- Pozitif eğim bize x arttıkça f(x)'nin de arttığını söyler. Negatif eğim bize x arttıkça f(x)'in azaldığını söyler. Sıfır eğim bize özel bir şey söylemez: fonksiyon o noktada artar, ne azalır veya yerel maksimumda veya yerel minimumda olabilir.

Türevler açısından bu bilgileri yazarken şunu görüyoruz:

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- $\frac{df(p)}{dx} > 0,$ ise f(x), x = p'de artan bir fonksiyondur.

- $\frac{df(p)}{dx} < 0,$ ise f(x), x = p'de azalan bir fonksiyondur.

- $\frac{df(p)}{dx} = 0,$ ise o zaman x = p, f (x)'in kritik noktası olarak adlandırılır ve x(p)'deki f (x) 'nin davranışı hakkında yorum yapılamaz.

Türevin Yorumu

İkinci Türev

- Bir fonksiyonun ikinci türevi, f'' (x) veya $\frac{d^2f}{dx^2}$ olarak yazılır. İlk türev bize fonksiyonun arttığını veya azaldığını söylese de, ikinci türev,

- x = p'de $\frac{d^2f(p)}{dx^2}$ >0 ise, f(x), x = p'de yukarı doğru kavislidir.

- x = p'de $\frac{d^2f(p)}{dx^2}$ <0 ise, f(x), x = p'de aşağı doğru kavislidir.

- x = p'de $\frac{d^2f(p)}{dx^2}$ =0 ise, o zaman f(x) 'in x = p'deki davranışı hakkında bir yorum yapamıyoruz.

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- Birinci türevin anlamından x, f(x) fonksiyonunun kritik bir noktası olduğunda, o noktada fonksiyonun davranışı hakkında bir yorum yapabilmek için, x'in bölgesel maksimum veya bölgesel minimum olduğunu öğrenmek için genellikle işlevin ikinci türevi kullanılır.

Denge – Dönüm - Eyer Noktaları

- Bir f fonksiyonunun konvekslikten konkavlığa veya konkavlıktan konveksliğe geçtiği ve fonksiyonunun sürekli olduğu noktaya dönüm (büküm) noktası adı verilir.

- Denge, değişmeyen bir sistem durumudur.

- İki boyutlu bir uzayda kararlı denge: düğüm ve odak

- Bir sistemin dinamikleri bir diferansiyel denklem veya bir diferansiyel denklem sistemi ile tanımlanıyorsa, o zaman denge noktası bir türevi (tüm türevleri) sıfıra ayarlayarak tahmin edilebilir.

- Özdeğerlerin hiçbirinin gerçek kısmı yoksa denge noktası hiperboliktir.

- Tüm özdeğerlerin negatif gerçek kısmı varsa, denge kararlı bir denklemdir.

- En az birinin pozitif bir gerçek kısmı varsa, denge kararsız bir düğümdür.

- En az bir özdeğerin negatif gerçek kısmı varsa ve en az birinin pozitif gerçek kısmı varsa, denge bir eyer noktasıdır.

Denge – Dönüm - Eyer Noktaları
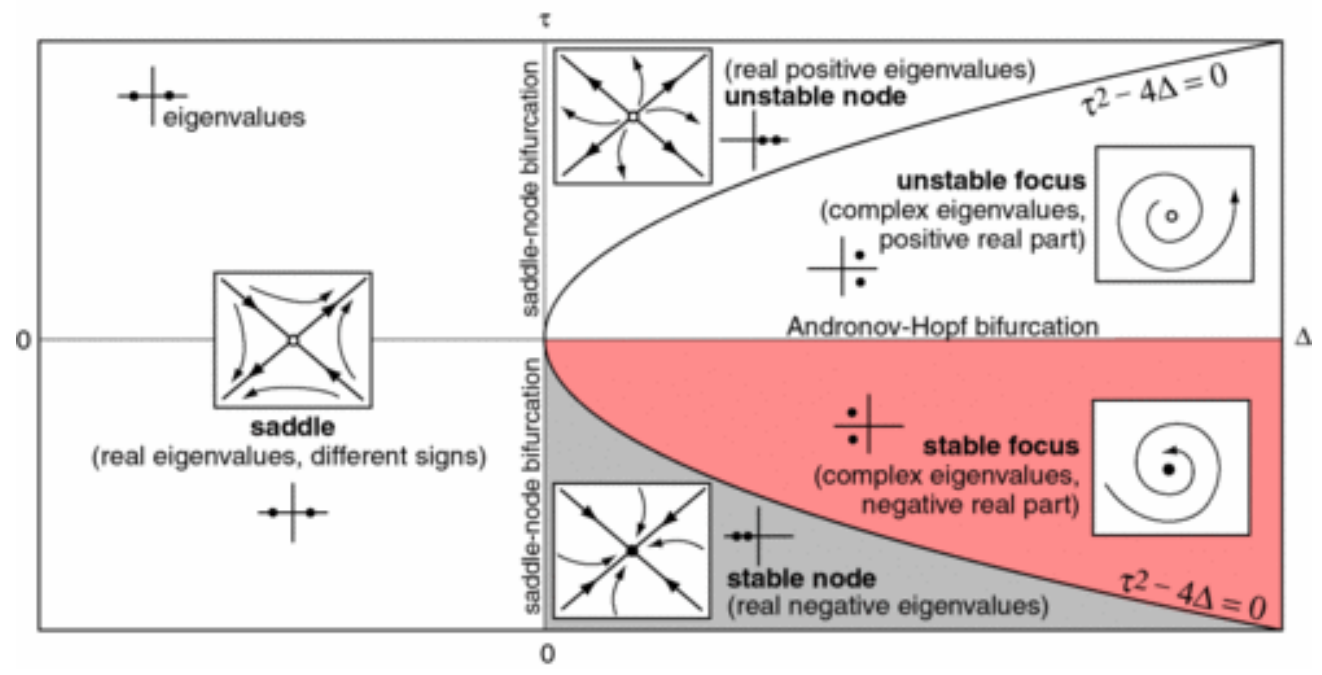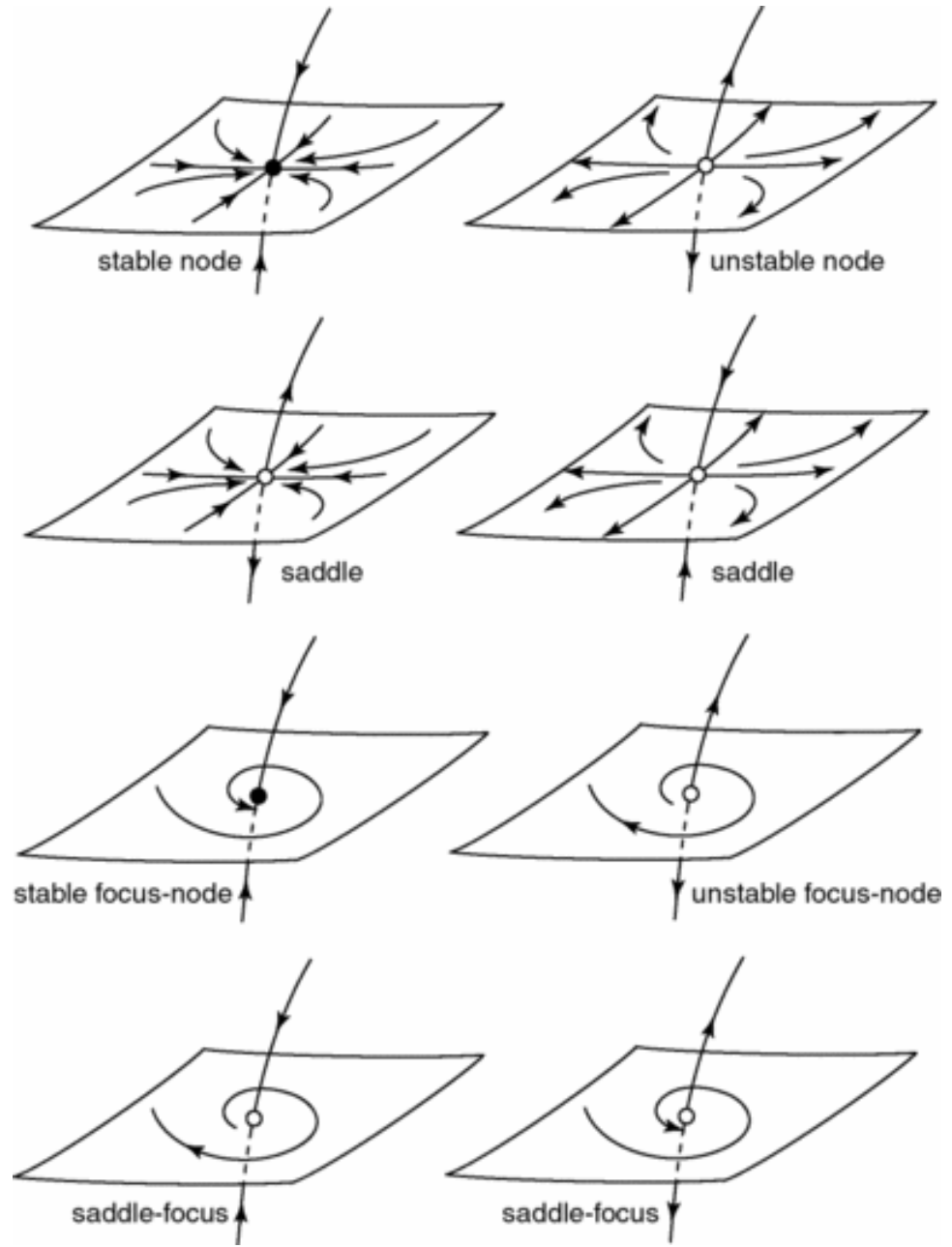
Belirsizlik

$\dfrac{\infty}{\infty}$ Belirsizlik Hali

Bu belirsizlik halinde de L' Hospital Kuralı geçerlidir. Zira $\quad\dfrac{u}{v}=\dfrac{1}{v}:\dfrac{1}{u}\quad$ biçiminde yazılabilir. Bu durumda $\dfrac{\infty}{\infty}$ belirsizliği $\dfrac{0}{0}$ belirsizliğe dönüşür.

- ## $0.\infty$ Belirsizlik Hali

  $u.v=\dfrac{u}{\frac{1}{v}}$ eşitliği yardımıyla $0.\infty$ belirsizliği $\dfrac{0}{0}$ veya $\dfrac{\infty}{\infty}$ haline getirilebilir.

Belirsizlik

- ## ∞-∞ Belirsizlik Hali

  Bu belirsizlik hali, $u - v = \dfrac{\frac{1}{v} - \frac{1}{u}}{\frac{1}{uv}}$ eşitliği yardımıyla $\dfrac{0}{0}$ belirsizlik haline dönüştürebilir.

- ## $0^0, \infty^0, 1^\infty$ Belirsizlik Halleri

  x sonlu bir değere veya $\pm\infty$ değerlerine yaklaştığında $y = [u(x)]^{v(x)}$ biçimindeki fonksiyonlar bu belirsizlik hallerinden birini verebilir. Bu durumda her iki tarafın logaritması alınarak

  $\ln y = v(x) \, In \, u(x)$

  eşitliği elde edilir. Sağdaki ifadenin limiti, $0.\infty$ belirsizliğine sahip olur. Bu limiti bilinen yolla hesaplanır.

  $\lim\limits_{x \to a} \ln y = \lambda \;\; ise \;\; \lim\limits_{x \to a} y = e^\lambda \;\; olur.$

Kavramlar

Dinamik Sistem

$$\dot{x} = f(x)$$

Lineer zamanla değişmeyen bir devrede elemanlardan en az bir tanesi endüktans, kapasite gibi enerji depolayan eleman ise o devreye **lineer zamanla değişmeyen dinamik devre** denir.

Denge Noktası

Lineer zamanla değişmeyen ve

$$\dot{x} = f(x)$$

bağıntısıyla ifade edilen bir sistemin denge noktaları

$$f(x) = 0$$

eşitliğinin reel kökleridir.

Özdeğer ve Özvektör

$$p(\lambda) = det(A - \lambda I) = 0$$

- Denklemini sağlayan λ değerlerine A matrisinin **özdeğer**leri denir.

- Bu λ değerini $Ax = \lambda x$ veya $(A - \lambda I)x = 0$

  da yerine koyarak elde edilen sıfırdan farklı çözümler, A matrisinin bu özdeğerine karşı gelen **özvektör**üdür.

Kararlılık nedir?

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Sistemdeki tüm bağımsız kaynakları devre dışı bırakalım, o halde çözüm:

$$x(t) = \phi(t).x(0)$$

Asimptotik Kararlılık

$$\dot{x} = Ax$$

Yukardaki denklemle tanımlı bir sistemimiz olsun.

- x=0, denge noktasında sistemimiz **kararlıdır,** ancak ve ancak ε>0 için öyle bir δ=δ(ε)>0 fonksiyonu olsun ki;
  ||x(0)||<δ ise ||x(t)||<ε,

- Kararlı değilse **kararsızdır.**

- Sistem kararlı ve ||x(0)||<δ koşuluna uyan bir δ varsa sistem aynı zamanda **asimptotik kararlıdır.**
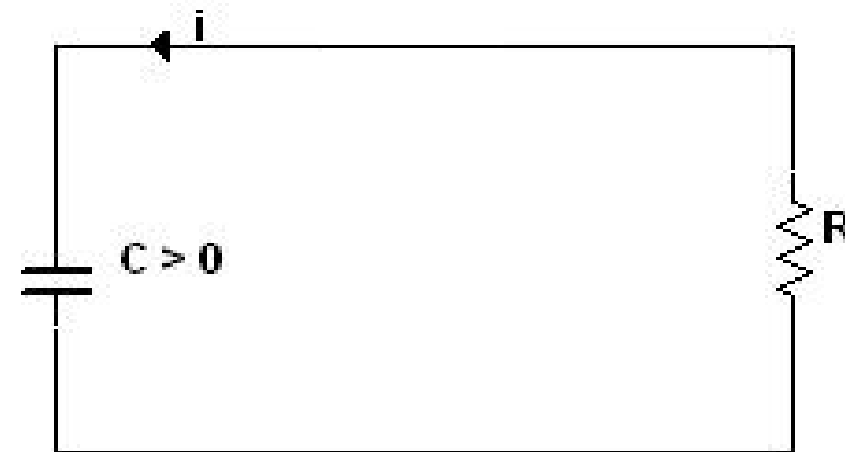
Örnek

$$Cx = -x/R$$

$$\dot{x} = -x/RC$$

Çözüm;

$$x(t) = e^{-t/RC}.x_0$$

- R > 0 ise sistem asimptotik kararlıdır.
- R = ∞ ise sistem kararlıdır (açık devre).
- R < 0 ise sistem kararsızdır.

Homogeneous Linear Systems

Homogeneous Linear Systems with Constant Coefficients

- We consider here a homogeneous system of n first order linear equations with constant, real coefficients:

$$x_1' = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n$$
$$\vdots$$
$$x_n' = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n$$

- This system can be written as **x′** = **Ax**, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Equilibrium Solutions

- Note that if $n = 1$, then the system reduces to

$$x' = ax \qquad \Longrightarrow \qquad x(t) = e^{at}$$

- Recall that $x = 0$ is the only equilibrium solution if $a \neq 0$.

- Further, $x = 0$ is an asymptotically stable solution if $a < 0$, since other solutions approach $x = 0$ in this case.

- Also, $x = 0$ is an unstable solution if $a > 0$, since other solutions depart from $x = 0$ in this case.

- For $n > 1$, equilibrium solutions are similarly found by solving $\mathbf{Ax} = \mathbf{0}$. We assume $\det\mathbf{A} \neq 0$, so that $\mathbf{x} = \mathbf{0}$ is the only solution. Determining whether $\mathbf{x} = \mathbf{0}$ is asymptotically stable or unstable is an important question here as well.

Phase Plane

- When $n = 2$, then the system reduces to

$$x_1' = a_{11}x_1 + a_{12}x_2$$

$$x_2' = a_{21}x_1 + a_{22}x_2$$

- This case can be visualized in the x_1x_2-plane, which is called the **phase plane**.

- In the phase plane, a direction field can be obtained by evaluating **Ax** at many points and plotting the resulting vectors, which will be tangent to solution vectors.

- A plot that shows representative solution trajectories is called a **phase portrait**.

- Examples of phase planes, directions fields and phase portraits will be given later in this section.

Solving Homogeneous System

- To construct a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, assume a solution of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, where the exponent r and the constant vector $\boldsymbol{\xi}$ are to be determined.

- Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ into $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain

$$r\boldsymbol{\xi}e^{rt} = \mathbf{A}\boldsymbol{\xi}e^{rt} \quad \Leftrightarrow \quad r\boldsymbol{\xi} = \mathbf{A}\boldsymbol{\xi} \quad \Leftrightarrow \quad (\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$$

- Thus to solve the homogeneous system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we must find the eigenvalues and eigenvectors of $\mathbf{A}$.

- Therefore $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ provided that r is an eigenvalue and $\boldsymbol{\xi}$ is an eigenvector of the coefficient matrix $\mathbf{A}$.

- Consider the homogeneous equation **x'** = **Ax** below.
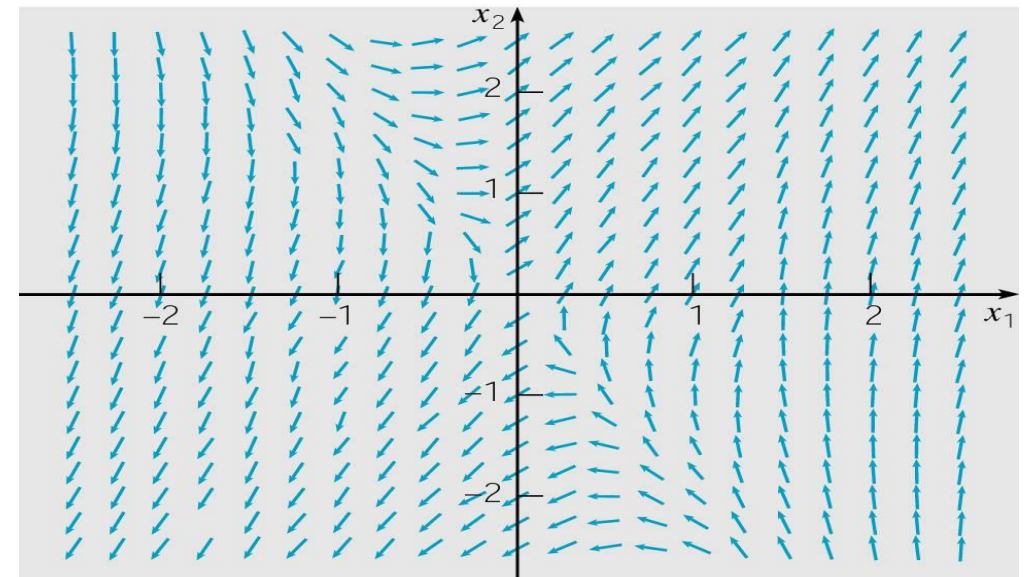
$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

- A direction field for this system is given below.

- Substituting **x** = **ξ**e^{rt} in for **x**, and rewriting system as (**A**-r**I**)**ξ** = **0**, we obtain

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Our solution has the form **x** = ξe^{rt}, where r and ξ are found by solving

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Recalling that this is an eigenvalue problem, we determine r by solving det(**A**-r**I**) = 0:

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = (r-3)(r+1)$$

- Thus r_1 = 3 and r_2 = -1.

- Eigenvector for $r_1 = 3$: Solve

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \iff \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -2 & 1 & 0 \\ 4 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 0 \\ 4 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{array}{l} 1\xi_1 & -1/2\xi_2 & = 0 \\ & 0\xi_2 & = 0 \end{array}$$

$$\rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1/2\xi_2 \\ \xi_2 \end{pmatrix} = c\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \quad c \text{ arbitrary} \rightarrow \text{choose } \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- Eigenvector for $r_2 = -1$: Solve

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \iff \begin{pmatrix} 1+1 & 1 \\ 4 & 1+1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 & 0 \\ 4 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{array}{r} 1\xi_1 + 1/2\xi_2 = 0 \\ 0\xi_2 = 0 \end{array}$$

$$\to \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -1/2\xi_2 \\ \xi_2 \end{pmatrix} = c\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, \quad c \text{ arbitrary} \to \text{choose } \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- The corresponding solutions $\mathbf{x} = \xi e^{rt}$ of $\mathbf{x}' = \mathbf{Ax}$ are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- The Wronskian of these two solutions is

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{-2t} \neq 0$$

- Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are fundamental solutions, and the general solution of $\mathbf{x}' = \mathbf{Ax}$ is

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- To visualize solution, consider first $\mathbf{x} = c_1\mathbf{x}^{(1)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} \quad\Leftrightarrow\quad x_1 = c_1e^{3t}, \quad x_2 = 2c_1e^{3t}$$

- Now

$$x_1 = c_1e^{3t}, \quad x_2 = 2c_1e^{3t} \quad\Leftrightarrow\quad e^{3t} = \frac{x_1}{c_1} = \frac{x_2}{2c_1} \quad\Leftrightarrow\quad x_2 = 2x_1$$

- Thus $\mathbf{x}^{(1)}$ lies along the straight line $x_2 = 2x_1$, which is the line through origin in direction of first eigenvector $\xi^{(1)}$

- If solution is trajectory of particle, with position given by

  (x_1, x_2), then it is in Q1 when $c_1 > 0$, and in Q3 when $c_1 < 0$.

- In either case, particle moves away from origin as t increases.

- Next, consider $\mathbf{x} = c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \quad \Leftrightarrow \quad x_1 = c_2 e^{-t}, \quad x_2 = -2c_2 e^{-t}$$

- Then $\mathbf{x}^{(2)}$ lies along the straight line $x_2 = -2x_1$, which is the line through origin in direction of 2nd eigenvector $\xi^{(2)}$

- If solution is trajectory of particle, with position given by (x_1, x_2), then it is in Q4 when $c_2 > 0$, and in Q2 when $c_2 < 0$.

- In either case, particle moves towards origin as t increases.

Example 1:
Phase Plane for General Solution

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$$

- As $t \to \infty$, $c_1\mathbf{x}^{(1)}$ is dominant and $c_2\mathbf{x}^{(2)}$ becomes negligible. Thus, for $c_1 \neq 0$, all solutions asymptotically approach the line $x_2 = 2x_1$ as $t \to \infty$.

- Similarly, for $c_2 \neq 0$, all solutions asymptotically approach the line $x_2 = -2x_1$ as $t \to -\infty$.

- The origin is a **saddle point**,
and is unstable. See graph.
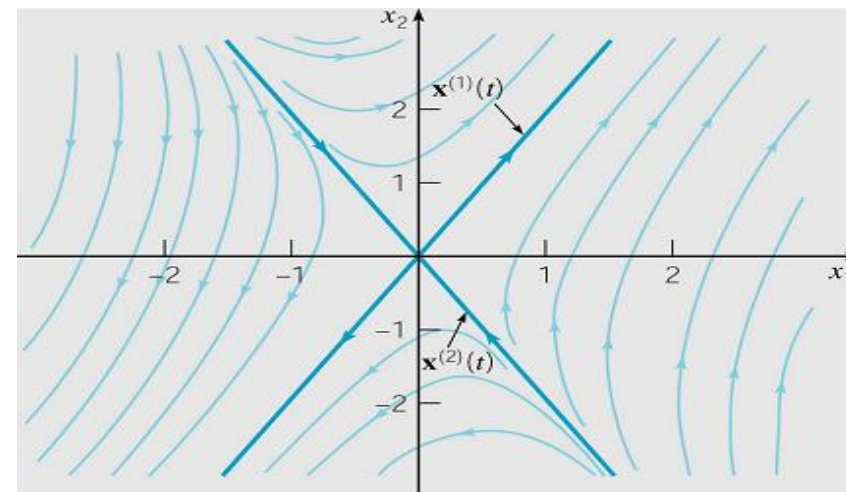
Example 1:
Time Plots for General Solution (9 of 9)

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \iff \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{pmatrix}$$

- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t. A few plots of x_1 are given below.

- Note that when $c_1 = 0$, $x_1(t) = c_2 e^{-t} \rightarrow 0$ as $t \rightarrow \infty$. Otherwise, $x_1(t) = c_1 e^{3t} + c_2 e^{t}$ grows unbounded as $t \rightarrow \infty$.

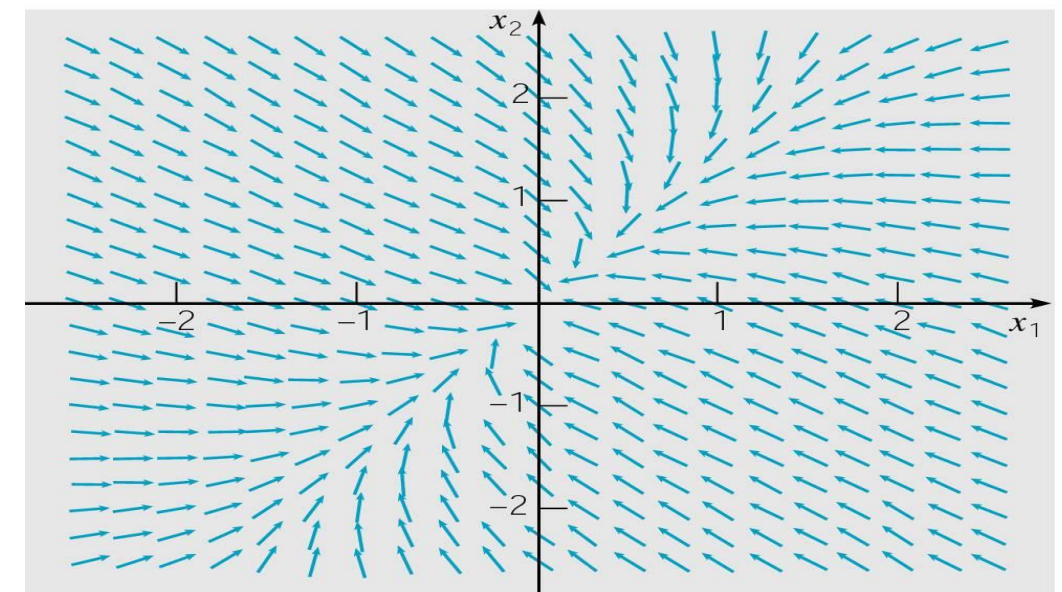- Graphs of x_2 are similarly obtained.
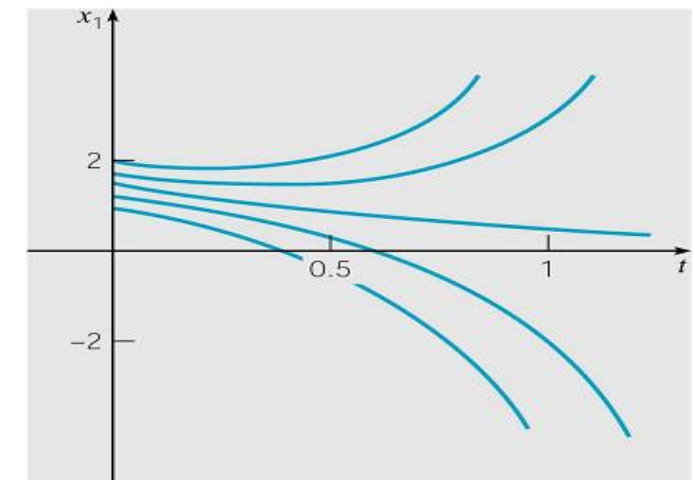
- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}$$

- Substituting $\mathbf{x} = \xi e^{rt}$ in for $\mathbf{x}$, and rewriting system as $(\mathbf{A}-r\mathbf{I})\xi = \mathbf{0}$, we obtain

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Our solution has the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$, where r and $\boldsymbol{\xi}$ are found by solving

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Recalling that this is an eigenvalue problem, we determine r by solving $\det(\mathbf{A}-r\mathbf{I}) = 0$:

$$\begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = (-3-r)(-2-r) - 2 = r^2 + 5r + 4 = (r+1)(r+4)$$

- Thus $r_1 = -1$ and $r_2 = -4$.

- Eigenvector for $r_1 = $ -1: Solve

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \iff \begin{pmatrix} -3+1 & \sqrt{2} \\ \sqrt{2} & -2+1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \sqrt{2}/2\xi_2 \\ \xi_2 \end{pmatrix} \rightarrow \text{choose } \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

- Eigenvector for $r_2 = -4$: Solve

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \iff \begin{pmatrix} -3+4 & \sqrt{2} \\ \sqrt{2} & -2+4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -\sqrt{2}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$\to \text{choose } \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

- The corresponding solutions $\mathbf{x} = \boldsymbol{\xi}e^{rt}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- The Wronskian of these two solutions is

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right](t) = \begin{vmatrix} e^{-t} & -\sqrt{2}e^{-4t} \\ \sqrt{2}e^{-t} & e^{-4t} \end{vmatrix} = 3e^{-5t} \neq 0$$

- Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are fundamental solutions, and the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- To visualize solution, consider first $\mathbf{x} = c_1\mathbf{x}^{(1)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}e^{-t} \quad \Leftrightarrow \quad x_1 = c_1e^{-t}, \quad x_2 = \sqrt{2}c_1e^{-t}$$

- Now

$$x_1 = c_1e^{-t}, \quad x_2 = \sqrt{2}c_1e^{-t} \quad \Leftrightarrow \quad e^{-t} = \frac{x_1}{c_1} = \frac{x_2}{\sqrt{2}c_1} \quad \Leftrightarrow \quad x_2 = \sqrt{2}x_1$$

- Thus $\mathbf{x}^{(1)}$ lies along the straight line $x_2 = 2^{\frac{1}{2}}x_1$, which is the line through origin in direction of first eigenvector $\xi^{(1)}$

- If solution is trajectory of particle, with position given by (x_1, x_2), then it is in Q1 when $c_1 > 0$, and in Q3 when $c_1 < 0$.

- In either case, particle moves towards origin as t increases.

- Next, consider $\mathbf{x} = c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \quad \Leftrightarrow \quad x_1 = -\sqrt{2}c_2 e^{-4t}, \quad x_2 = c_2 e^{-4t}$$

- Then $\mathbf{x}^{(2)}$ lies along the straight line $x_2 = -2^{\frac{1}{2}}x_1$, which is the line through origin in direction of 2nd eigenvector $\xi^{(2)}$

- If solution is trajectory of particle, with position given by

  (x_1, x_2), then it is in Q4 when $c_2 > 0$, and in Q2 when $c_2 < 0$.

- In either case, particle moves towards origin as t increases.

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- As $t \rightarrow \infty$, $c_1\mathbf{x}^{(1)}$ is dominant and $c_2\mathbf{x}^{(2)}$ becomes negligible. Thus, for $c_1 \neq 0$, all solutions asymptotically approach origin along the line $x_2 = 2^{\frac{1}{2}} x_1$ as $t \rightarrow \infty$.

- Similarly, all solutions are unbounded as $t \rightarrow -\infty$.

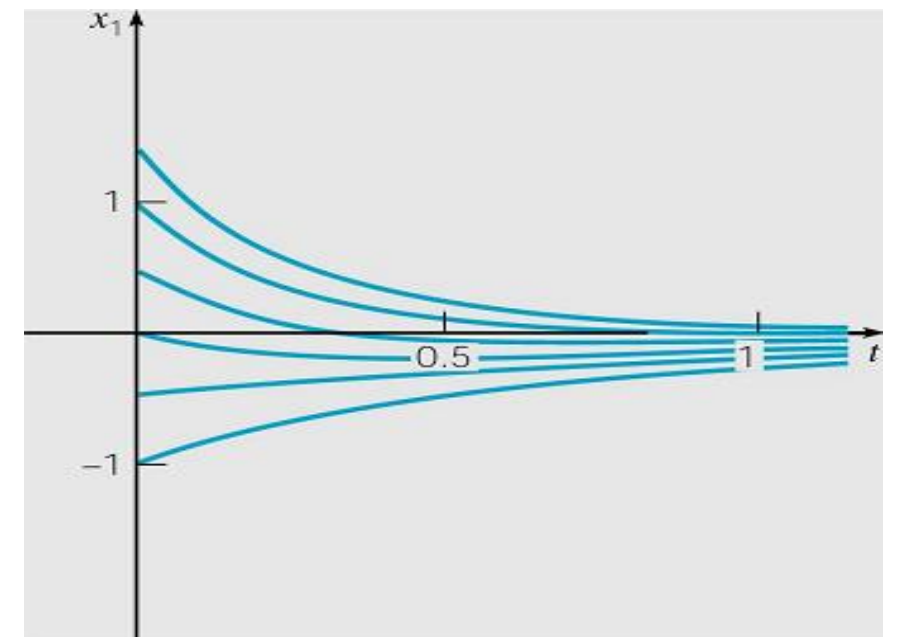- The origin is a **node**, and is asymptotically stable.
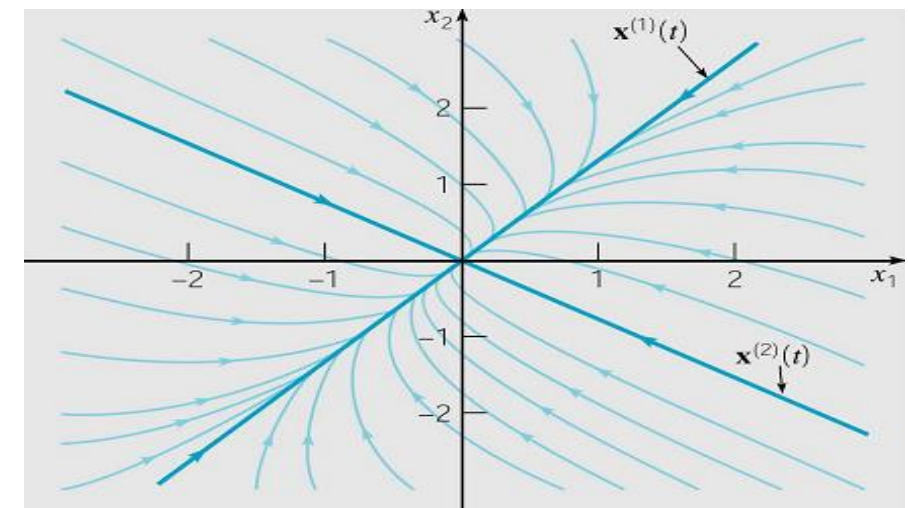
- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}e^{-t} + c_2\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}e^{-4t} \iff \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1e^{-t} - \sqrt{2}c_2e^{-4t} \\ \sqrt{2}c_1e^{-t} + c_2e^{-4t} \end{pmatrix}$$

- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t. A few plots of x_1 are given below.

- Graphs of x_2 are similarly obtained.

2 x 2 Case:
Real Eigenvalues, Saddle Points and Nodes

- The previous two examples demonstrate the two main cases for a 2 x 2 real system with real and different eigenvalues:

  - Both eigenvalues have opposite signs, in which case origin is a saddle point and is unstable.

  - Both eigenvalues have the same sign, in which case origin is a node, and is asymptotically stable if the eigenvalues are negative and unstable if the eigenvalues are positive.
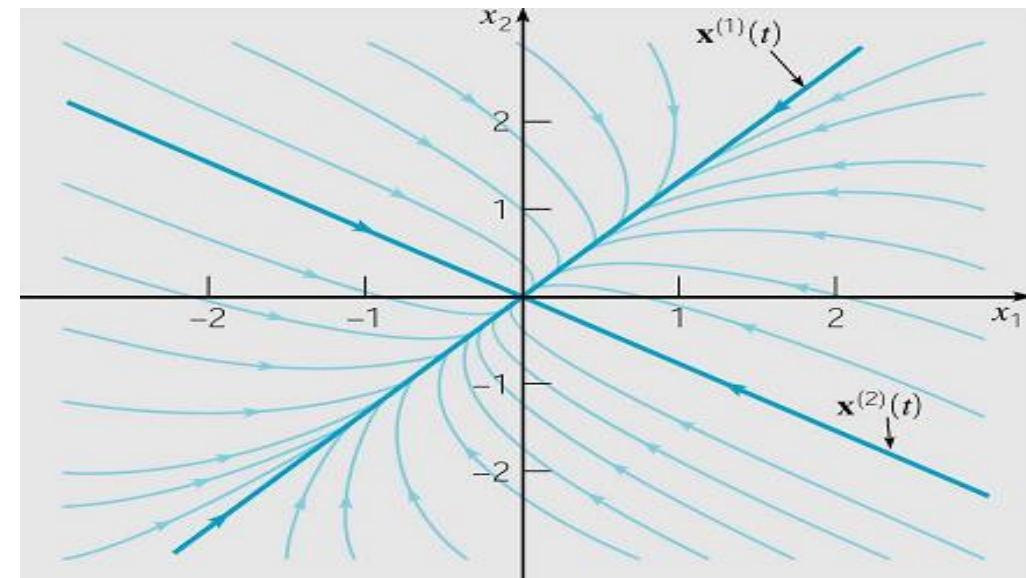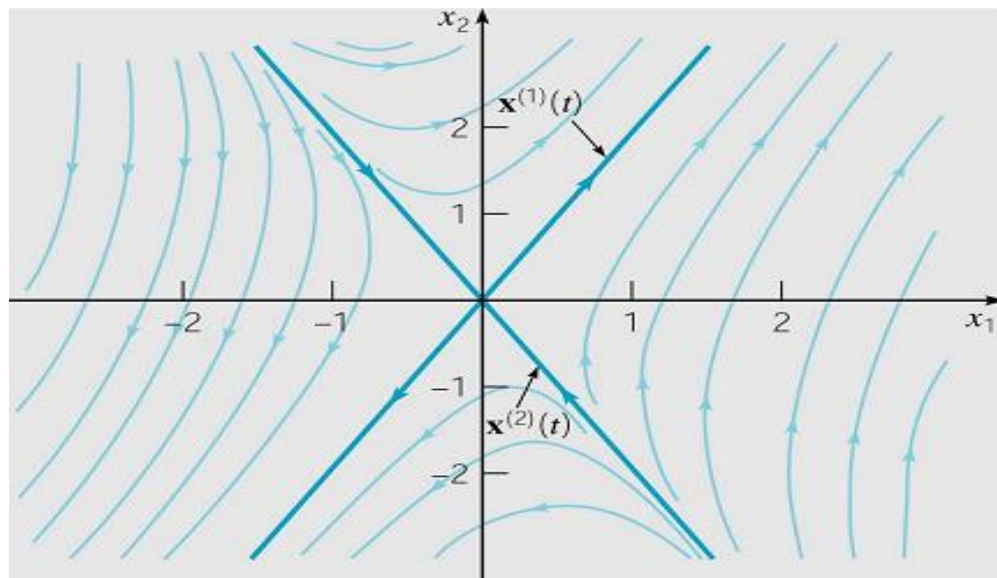
Eigenvalues, Eigenvectors and Fundamental Solutions

- In general, for an n x n real linear system $\mathbf{x}' = \mathbf{Ax}$:
  - All eigenvalues are real and different from each other.
  - Some eigenvalues occur in complex conjugate pairs.
  - Some eigenvalues are repeated.
- If eigenvalues $r_1, ..., r_n$ are real & different, then there are n corresponding linearly independent eigenvectors $\xi^{(1)}, ..., \xi^{(n)}$. The associated solutions of $\mathbf{x}' = \mathbf{Ax}$ are

$$\mathbf{x}^{(1)}(t) = \xi^{(1)}e^{r_1 t}, ..., \mathbf{x}^{(n)}(t) = \xi^{(n)}e^{r_n t}$$

- Using Wronskian, it can be shown that these solutions are linearly independent, and hence form a fundamental set of solutions. Thus general solution is

$$\mathbf{x} = c_1\xi^{(1)}e^{r_1 t} + ... + c_n\xi^{(n)}e^{r_n t}$$

Hermitian Case: Eigenvalues, Eigenvectors & Fundamental Solutions

- If **A** is an *n* x *n* Hermitian matrix (real and symmetric), then all eigenvalues $r_1, \ldots, r_n$ are real, although some may repeat.

- In any case, there are *n* corresponding linearly independent and orthogonal eigenvectors $\xi^{(1)}, \ldots, \xi^{(n)}$. The associated solutions of **x'** = **Ax** are

$$\mathbf{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \xi^{(n)} e^{r_n t}$$

and form a fundamental set of solutions.

- Consider the homogeneous equation **x'** = **Ax** below.

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$$

- The eigenvalues were found previously in Ch 7.3, and were: $r_1 = 2$, $r_2 = -1$ and $r_3 = -1$.

- Corresponding eigenvectors:

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \; \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

- The fundamental solutions are

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \ \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

with general solution

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)}$:

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

- As $t \rightarrow \infty$, $c_1\mathbf{x}^{(1)}$ is dominant and $c_2\mathbf{x}^{(2)}$, $c_3\mathbf{x}^{(3)}$ become negligible.

- Thus, for $c_1 \neq 0$, all solns $\mathbf{x}$ become unbounded as $t \rightarrow \infty$,

  while for $c_1 = 0$, all solns $\mathbf{x} \rightarrow \mathbf{0}$ as $t \rightarrow \infty$.

- The initial points that cause $c_1 = 0$ are those that lie in plane determined by $\xi^{(2)}$ and $\xi^{(3)}$. Thus solutions that start in this plane approach origin as $t \rightarrow \infty$.

Complex Eigenvalues and Fundamental Solns

- If some of the eigenvalues $r_1,\ldots, r_n$ occur in complex conjugate pairs, but otherwise are different, then there are still n corresponding linearly independent solutions

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t},\ldots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t},$$

which form a fundamental set of solutions. Some may be complex-valued, but real-valued solutions may be derived from them.

- If the coefficient matrix **A** is complex, then complex eigenvalues need not occur in conjugate pairs, but solutions will still have the above form (if the eigenvalues are distinct) and these solutions may be complex-valued.

Repeated Eigenvalues and Fundamental Solns

- If some of the eigenvalues $r_1,\ldots, r_n$ are repeated, then there may not be n corresponding linearly independent solutions of the form

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t},\ldots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t}$$

- In order to obtain a fundamental set of solutions, it may be necessary to seek additional solutions of another form.

- This situation is analogous to that for an nth order linear equation with constant coefficients, in which case a repeated root gave rise solutions of the form

  This case of repeated eigenvalues is examined in Section 7.8.

$$e^{rt},te^{rt},t^2 e^{rt},\ldots$$